Coloring problems

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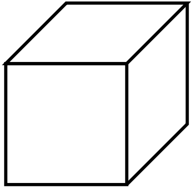

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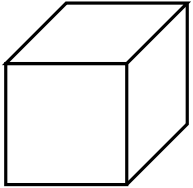
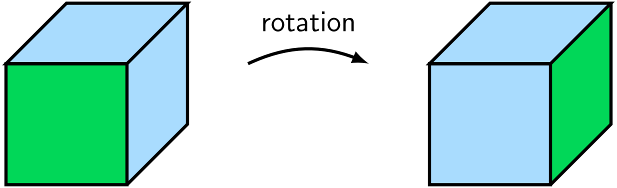


$\rightsquigarrow$ $2^6 = 64$ colorings of the cube.

... but a non trivial answer !



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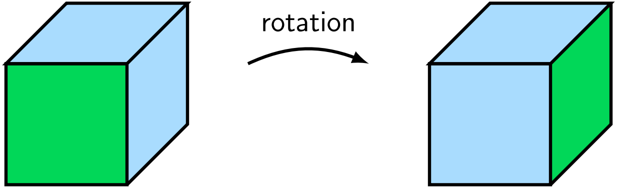

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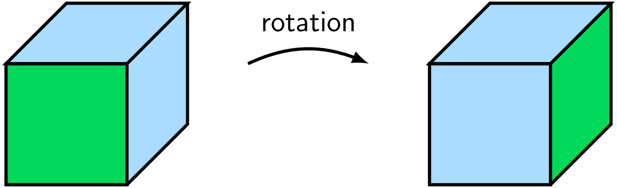

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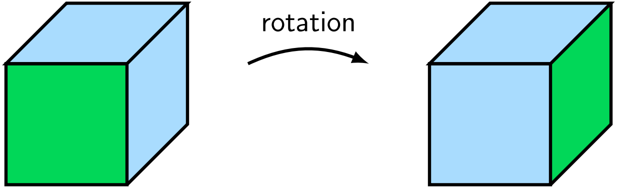

Now, we identify two colorings whenever one can be transformed into the other by manipulating the object.

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$\rightsquigarrow$ The answer is less obvious and requires the use of group actions.

q-colorings

Definition (q-coloring)

Given a nonempty finite set E and a positive integer q, a q-**coloring** of E is a map from E to $\{1, ..., q\}$.

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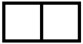
Example. Consider a domino , and assume that blue and green are the only available colors.

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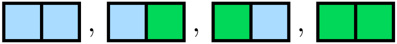
q-colorings

Thus

$$C(E,q) = \left\{ \begin{array}{ccc} E & \mapsto & \{1,2\} \\ A & \mapsto & 1 \\ B & \mapsto & 1 \end{array}, \begin{array}{ccc} E & \mapsto & \{1,2\} \\ A & \mapsto & 1 \\ B & \mapsto & 2 \end{array}, \begin{array}{ccc} E & \mapsto & \{1,2\} \\ A & \mapsto & 2 \\ B & \mapsto & 1 \end{array}, \begin{array}{ccc} E & \mapsto & \{1,2\} \\ A & \mapsto & 2 \\ B & \mapsto & 2 \end{array} \right\}$$

"=" {  }.

The group action

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Notation. For every $g \in G$ and for every $f \in C(E, q)$, we denote by $g \cdot f$ the element of $C(E, q)$ defined for every $x \in E$ by

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Proposition

The map $\begin{array}{ccc} G \times C(E,q) & \to & C(E,q) \\ (g,f) & \mapsto & g \cdot f \end{array}$ is a group action.

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Proof. Let $f \in C(E, q)$. For every $x \in E$,

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Let $g \in G$, $g' \in G$ and $f \in C(E,q)$. For every $x \in E$,

$$\begin{aligned}(g \cdot (g' \cdot f))(x) &= (g' \cdot f)(g^{-1} \cdot x) = f(g'^{-1} \cdot (g^{-1} \cdot x)) \\ &= f((g'^{-1} g^{-1}) \cdot x) = f((gg')^{-1} \cdot x) \\ &= ((gg') \cdot f)(x).\end{aligned}$$

Hence $(gg') \cdot f = g \cdot (g' \cdot f)$. ❑

Equivalent colorings

Definition (equivalent colorings)

Let E be a nonempty finite set, q a positive integer and G a finite group acting on E.

We say that two colorings $f \in C(E, q)$ and $f' \in C(E, q)$ are **equivalent** if there exists $g \in G$ such that $f' = g \cdot f$ (that is, f and f' belong to the same orbit).

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$\rightsquigarrow$ **Goal**. To determine $|C_G(E, q)|$.

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Example. Let us go back to the domino $\boxed{}\!\boxed{}$. As before, $E = \{A, B\}$ "=" $\boxed{A}\!\boxed{B}$, and $\{1, 2\}$ "=" $\{$blue, green$\}$.

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Maps $f_1 : \begin{array}{ccc} E & \to & \{1,2\} \\ A & \mapsto & 1 \\ B & \mapsto & 1 \end{array}$ and $f_2 : \begin{array}{ccc} E & \to & \{1,2\} \\ A & \mapsto & 2 \\ B & \mapsto & 2 \end{array}$ are not equivalent, but $f_3 : \begin{array}{ccc} E & \to & \{1,2\} \\ A & \mapsto & 1 \\ B & \mapsto & 2 \end{array}$ and $f_4 : \begin{array}{ccc} E & \to & \{1,2\} \\ A & \mapsto & 2 \\ B & \mapsto & 1 \end{array}$ are equivalent (because $f_3 = \tau \cdot f_4$).

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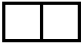
Example. Let us go back to the domino . As before, $E = \{A, B\}$ "=" $\boxed{A \mid B}$, and $\{1, 2\}$ "=" $\{$blue, green$\}$.

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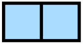
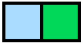
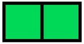
$$C_G(E, q) = \{[f_1], [f_2], [f_3]\} \text{ "=" } \Big\{$$ , ,  $$\Big\}.$$

Burnside's lemma

Proposition (Burnside's lemma)

Let G be a finite group acting on a nonempty set X. Let Ω denote the set of orbits of the action.

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Corollary

With the notation of the previous proposition, we have

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Characterization of $C(E, q)(g)$ in terms of g-orbits

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If f is constant on each g-orbit, then for all $x \in E$, x and $g^{-1} \cdot x$ lie in the same g-orbit, so $f(g^{-1} \cdot x) = f(x)$. Hence, $g \cdot f = f$. ❑

A formula for $|C_G(E,q)|$

Theorem

$$|C_G(E,q)| = \frac{1}{|G|}\sum_{g\in G} q^{r(g)}$$

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Proof. By Burnside's lemma, $|C_G(E, q)| = \dfrac{1}{|G|} \sum_{g \in G} |C(E, q)(g)|$.

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Fix $g \in G$ and $f \in C(E,q)$. We know that $f \in C(E,q)(g)$ if and only if f is constant on each g-orbit. For each g-orbit, we have q possible choices of colors. Therefore,

$$|C_G(E,q)(g)| = \underbrace{q \times \ldots \times q}_{r(g)\text{ times}} = q^{r(g)}.$$

$\square$

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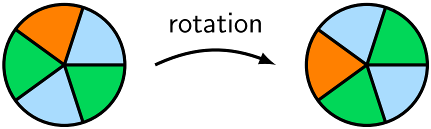

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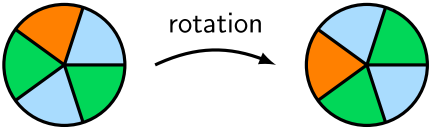

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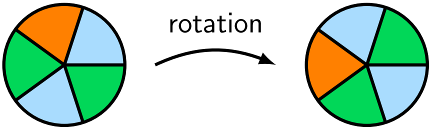

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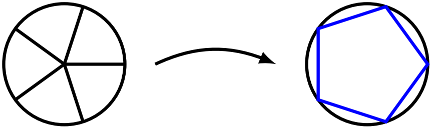
$\rightsquigarrow$ **Idea**. Identify sectors of the wheel with vertices of a regular n-gon.



Reformulation via group actions

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Fix $g \in G$. A g-orbit is an orbit under the action $\langle g \rangle \curvearrowright E$. First, $|\langle g \rangle| = o(g)$. Second, for all $S \in E$, $|\mathrm{Stab}_{\langle g \rangle}(S)| = 1$.

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$$r(g) = \frac{1}{|\langle g \rangle|} \sum_{S \in E} |\mathrm{Stab}_{\langle g \rangle}(S)| = \frac{1}{o(g)} \sum_{S \in E} 1 = \frac{n}{o(g)}.$$

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However, by Lagrange's theorem, for every $g \in G$, $o(g)$ divides n. For each positive divisor d of n, the number of elements of order d in G is equal to $\varphi(d)$ where

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Therefore,

$$\boxed{|C_G(E,q)| = \frac{1}{n}\sum_{\substack{d|n \\ d>0}} \varphi(d) q^{\frac{n}{d}}.}$$

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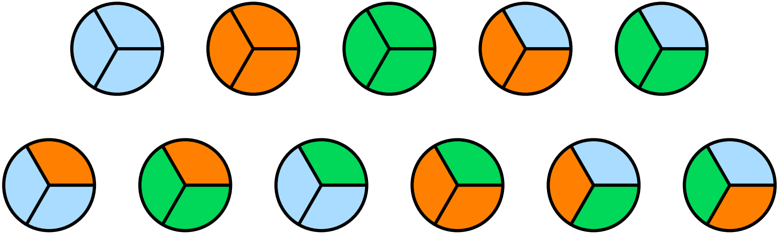

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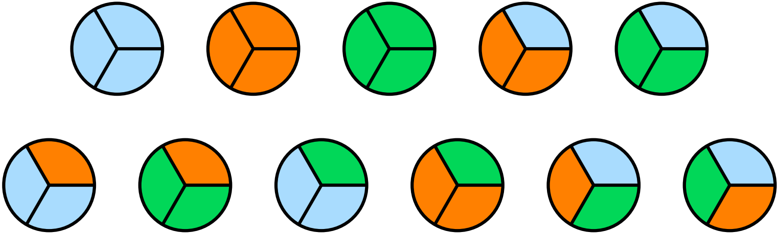

With the previous formula, we have

$$|C_G(E,q)| = \frac{1}{3}\sum_{\substack{d|3\\d>0}} \varphi(d)3^{\frac{3}{d}} = \frac{1}{3}(3^3+6) = 11.$$

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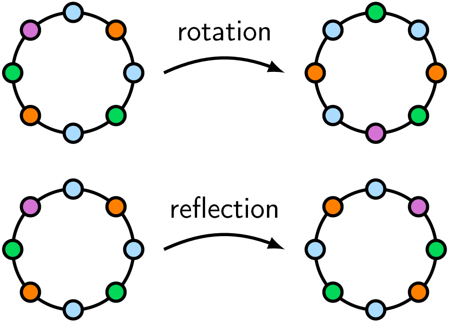

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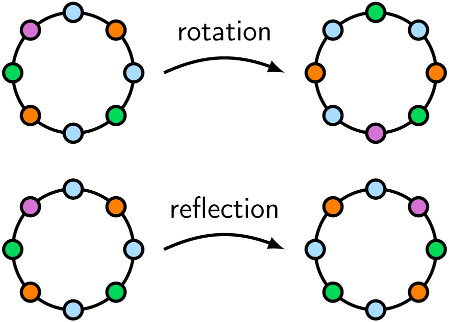
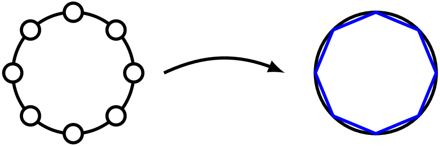


Goal. Determine the number of distinct colorings.

Reformulation via group actions

⇝ **Idea**. Identify beads of the necklace with vertices of a regular n-gon.



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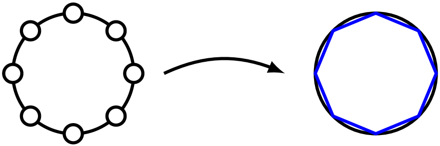

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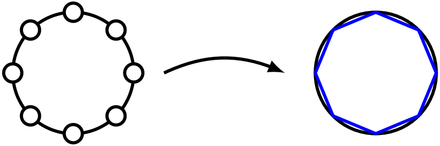

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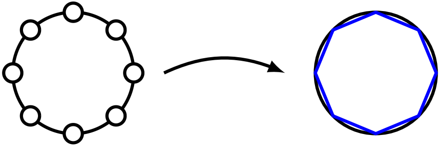

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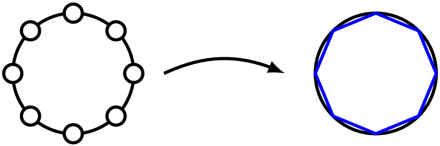

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The situation is different for reflections.

Case 1 : n is odd

Suppose that $n = 2m - 1$ for some positive integer m.

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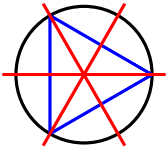

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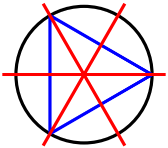

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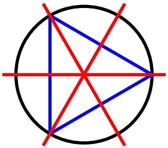

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$$\boxed{\text{If } n \text{ is odd, } |C_G(E,q)| = \frac{1}{2n}\left(\sum_{\substack{d|n\\ d>0}} \varphi(d) q^{\frac{n}{d}} + n q^{\frac{n+1}{2}}\right).}$$

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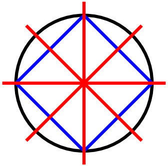

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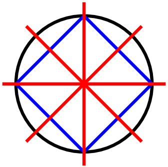

Thus, we have $m = n/2$ reflections ; each having two orbits of size 1 and $m - 1$ orbits of size 2, that is, a total of $m + 1 = (n+2)/2$ orbits ; and $m = n/2$ reflections, each having m orbits of size 2.

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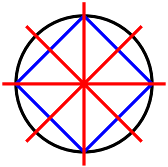

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An example

If $n = 4$ and $q = 2$, we have 6 distinct colorings of the necklace.

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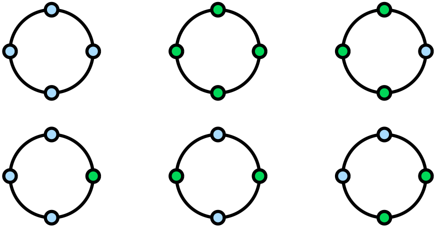

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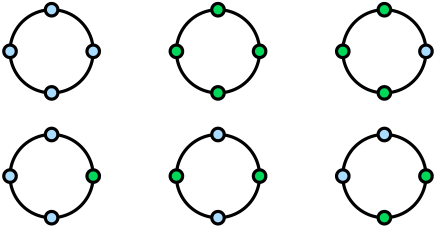

With the previous formula, we have

$$|C_G(E,q)| = \frac{1}{8}\left(\sum_{\substack{d|4\\ d>0}} \varphi(d)2^{\frac{4}{d}} + 2\times 2^3 + 2\times 2^2\right)$$
$$= \frac{1}{8}(16+4+4+16+8) = 6.$$

Thank you for your attention !